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LETTER TO THE EDITOR

Non-perturbatively defined non-local currents for restricted conformal $\widehat{sl}(2)$ Toda model*

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Abstract. The non-local conserved currents for the restricted quantum conformal $\widehat{sl}(2)$ Toda field system are obtained non-perturbatively, and the relations between these results and the perturbative BLs are discussed.

Integrable perturbative theories have played important roles in the development of two-dimensional conformal field theories. Since the pioneering work of Zamolodchikov [1], much progress has been made in this direction. Up to now, one of the most important developments has perhaps been the non-local conserved charge approach advocated by Bernard and LeClair [2], which is a way to match the famous quantum inverse scattering method. According to the BL approach, the Toda-like dynamic system under consideration should be formulated in the perturbed CFT framework at first, then their hidden quantum group symmetries and integrabilities can be revealed by their four non-trivially defined non-local conserved charges, as well as the commutation relations satisfied by these charges. So far, the BL method has successfully been applied for the Sine–Gordon and affine Toda models [2], conformal $\widehat{sl}(2)$ Toda model [2], Thirring model [3], ZMS model [4] and various completely integrable perturbation systems of superconformal theories [5–7]. The factorizable S matrices of these systems have been obtained.

It is worth stressing that in perturbed CFTs, although the expressions for non-local conserved charges are approximate, their commutation algebra and the dependence on coupling λ are non-perturbative. It is then natural to pursue the non-perturbative versions of such non-local conserved charges. Recently, Chang and Rajaraman [8] have made some enlightening work on these lines. They have proposed a non-perturbative counterpart of the BL approach. Such a non-perturbative approach is mostly successful for the Sine–Gordon model, which yields the same charge algebras and the same soliton S-matrix as those given by BL except that there are some differences in the expressions of non-local conserved charges. The difference of the charges between the two methods comes from the fact that one is perturbative and another is non-perturbative. It is more important and more interesting to explore whether the non-local currents and charges in other integrable models can also be non-perturbatively defined. We will study the $\widehat{sl}(2)$ Toda model in this paper. We find that the CR method can be successfully applied to such a model as well

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as the simply laced affine Toda and conformal affine Toda systems, but this method is not suitable for the non-simply-laced affine Toda and non-simply-laced conformal affine Toda models. In the next section, we will discuss the canonical quantization of the system. The third section gives out the chiral field transformation of our model. In the fourth section, the non-local conserved currents will be defined non-perturbatively. Finally, we will make some remarks.

Canonical quantization

We consider a conformal $\widehat{sl}(2)$ Toda model [9]. Its action is

$$S = \frac{1}{4\pi}\int d^2x\left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \partial_\mu\eta\partial^\mu\xi - 2\lambda\left(e^{i\beta\phi} + e^{i\beta\eta - i\beta\phi}\right)\right] \tag{1}$$

where β is a real coupling constant. Correspondingly, the equations of motion read

$$\begin{aligned}&\partial_\mu\partial^\mu\phi + 2i\lambda\beta\left(e^{i\beta\phi} - e^{i\beta\eta - i\beta\phi}\right) = 0\\ &\partial_\mu\partial^\mu\xi + 2i\lambda\beta e^{i\beta\eta - i\beta\phi} = 0\\ &\partial_\mu\partial^\mu\eta = 0\end{aligned} \tag{2}$$

which are an extension of the Sine–Gordon model.

The quantization of system (1) needs to set up its Hamiltonian formalism. We regard the fields $\phi(x)$, $\eta(x)$ and $\xi(x)$ as the canonical coordinates in the phase space and define their canonical conjugate momenta, as well as the equal-time Poisson brackets, as

$$\pi_\phi(x) = \frac{1}{4\pi}\partial_0\phi(x) \qquad \pi_\xi(x) = \frac{1}{4\pi}\partial_0\eta(x) \qquad \pi_\eta(x) = \frac{1}{4\pi}\partial_0\xi(x) \tag{3}$$

$$\{\phi(x), \pi_\phi(y)\} = \{\xi(x), \pi_\xi(y)\} = \{\eta(x), \pi_\eta(y)\} = \delta(x^1 - y^1)\,. \tag{4}$$

The corresponding canonical Hamiltonian H_c is

$$H_c = \frac{1}{4\pi}\int dx^1\left[\tfrac{1}{2}\left((\partial_0\phi)^2 + (\partial_1\phi)^2\right) + (\partial_0\eta)(\partial_0\xi) + (\partial_1\eta)(\partial_1\xi) + 2\lambda(e^{i\beta\phi} + e^{i\beta\eta - i\beta\phi})\right] \tag{5}$$

which is nothing but one of the generators of the Poincaré transformation.

In (1+1) spacetime the Poincaré group is a three-dimensional group. The other generators read

$$\begin{aligned}P &= \frac{1}{4\pi}\int dx^1[(\partial_0\phi)(\partial_1\phi) + (\partial_0\eta)(\partial_1\xi)]\\ M &= \frac{1}{4\pi}\int dx^1\left[x^0((\partial_0\phi)(\partial_1\phi) + (\partial_0\eta)(\partial_1\xi)) + x^1\left(\tfrac{1}{2}(\partial_0\phi)^2 + \tfrac{1}{2}(\partial_1\phi)^2\right.\right.\\ &\qquad \left.\left. + (\partial_0\eta)(\partial_0\xi) + (\partial_1\eta)(\partial_1\xi) + 2\lambda\left(e^{i\beta\phi} + e^{i\beta\eta - i\beta\phi}\right)\right)\right]\end{aligned} \tag{6}$$

which stand for the momentum and the angular momentum of this system, respectively.

In the canonical quantum theory of system (1), the above canonical variables and their functions, turn out to become the Hermitian operators in Hilbert space. The canonical quantization can be formally realized by making the replacement $\{A, B\} \to (1/i)[A, B]$, where $[A, B]$ is the equal-time commutator. Effecting this substitution in (4), we get

$$[\phi(x), \pi_\phi(y)] = [\xi(x), \pi_\xi(y)] = [\eta(x), \pi_\eta(y)] = i\delta\left(x^1 - y^1\right). \tag{7}$$

Moreover, (5) and (6) continue to hold in the sense of the normal ordering. To make this more transparent, we will give a detailed explanation for normal ordering in the present case. As the fields obey coupled nonlinear equations of motion, we cannot expand them in terms of plane waves as in the non-interaction case. Instead, these operators can be expanded in terms of their Fourier components at an arbitrary given time x^0:

$$\phi(x) = \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{|k|}}\left[a_k e^{ikx^1} + a_k^\dagger e^{-ikx^1}\right] \qquad \pi_\phi(x) = -\frac{i}{4\pi}\int_{-\infty}^{+\infty} dk\sqrt{|k|}\left[a_k e^{ikx^1} - a_k^\dagger e^{-ikx^1}\right]$$
$$\xi(x) = \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{|k|}}\left[b_k e^{ikx^1} + c_k^\dagger e^{-ikx^1}\right] \qquad \pi_\xi(x) = -\frac{i}{4\pi}\int_{-\infty}^{+\infty} dk\sqrt{|k|}\left[c_k e^{ikx^1} - b_k^\dagger e^{-ikx^1}\right] \quad (8)$$
$$\eta(x) = \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{|k|}}\left[c_k e^{ikx^1} + b_k^\dagger e^{-ikx^1}\right] \qquad \pi_\eta(x) = -\frac{i}{4\pi}\int_{-\infty}^{+\infty} dk\sqrt{|k|}\left[b_k e^{ikx^1} - c_k^\dagger e^{-ikx^1}\right].$$

The operators a_k, b_k, c_k and $a_k^\dagger, b_k^\dagger$ as well as $c_k^\dagger$ appearing in (8) are the corresponding soliton annihilation and creation operators, respectively, which satisfy the following commutation relations:

$$\left[a_k, a_{k'}^\dagger\right] = \left[b_k, b_{k'}^\dagger\right] = \left[c_k, c_{k'}^\dagger\right] = \delta(k-k')\,. \qquad (9)$$

An operator product is in normal ordered form if all creation operators stand to the left of all annihilation operators.

Chiral fields transformation

To define and evaluate non-perturbatively the non-local conserved currents for the quantum conformal $\widehat{sl}(2)$ Toda model, we feel obliged to decompose the canonical variables into their 'chiral' components:

$$\rho_\phi(x) = \tfrac{1}{2}\left[\phi(x) + 4\pi\int_{-\infty}^{x^1} dy^1\,\pi_\phi(y)\right] \qquad \bar\rho_\phi(x) = \tfrac{1}{2}\left[\phi(x) - 4\pi\int_{-\infty}^{x^1} dy^1\,\pi_\phi(y)\right]$$
$$\rho_\xi(x) = \tfrac{1}{2}\left[\xi(x) + 4\pi\int_{-\infty}^{x^1} dy^1\,\pi_\eta(y)\right] \qquad \bar\rho_\xi(x) = \tfrac{1}{2}\left[\xi(x) - 4\pi\int_{-\infty}^{x^1} dy^1\,\pi_\eta(y)\right] \quad (10)$$
$$\rho_\eta(x) = \tfrac{1}{2}\left[\eta(x) + 4\pi\int_{-\infty}^{x^1} dy^1\,\pi_\xi(y)\right] \qquad \bar\rho_\eta(x) = \tfrac{1}{2}\left[\eta(x) - 4\pi\int_{-\infty}^{x^1} dy^1\,\pi_\xi(y)\right].$$

As a matter of fact, only the $\rho_\xi(x)$ and $\bar\rho_\xi(x)$ are genuine chiral operators. The components $\rho_\phi(x)$, $\bar\rho_\phi(x)$, $\rho_\eta(x)$ and $\bar\rho_\eta(x)$ become chiral operators only when λ tends to zero, because

$$\partial_-\rho_\phi = -\partial_+\bar\rho_\phi = -i\beta\lambda\int_{-\infty}^{x^1} dy^1\left(e^{i\beta\phi} - e^{i\beta\eta - i\beta\phi}\right)$$
$$\partial_-\rho_\xi = -\partial_+\bar\rho_\xi = 0 \qquad (11)$$
$$\partial_-\rho_\eta = -\partial_+\bar\rho_\eta = -i\beta\lambda\int_{-\infty}^{x^1} dy^1 e^{i\beta\eta - i\beta\phi}$$

where $\partial_\pm = \partial_0 \pm \partial_1$. We call them 'chiral' operators only for convenience to compare with BLs. With these 'chiral' operators we can recast the generators of Poincaré transformation as follows:

$$H = \frac{1}{4\pi}\int dx^1\big[(\partial_1\rho_\phi)^2 + (\partial_1\bar{\rho}_\phi)^2 + 2(\partial_1\rho_\eta\partial_1\rho_\xi) + 2(\partial_1\bar{\rho}_\eta\partial_1\bar{\rho}_\xi) + 2\lambda(W_{\beta,\beta} + \tilde{W}_{\beta,\beta}W_{-\beta,-\beta})\big] \tag{12}$$

$$P = \frac{1}{4\pi}\int dx^1\big[(\partial_1\rho_\phi)^2 - (\partial_1\bar{\rho}_\phi)^2 + 2(\partial_1\rho_\xi\partial_1\rho_\eta) - 2(\partial_1\bar{\rho}_\xi\partial_1\bar{\rho}_\eta)\big] \tag{13}$$

$$M = \frac{1}{4\pi}\int dx^1\big[(x^1 + x^0)((\partial_1\rho_\phi)^2 + 2(\partial_1\rho_\xi\partial_1\rho_\eta)) + (x^1 - x^0)((\partial_1\bar{\rho}_\phi)^2 + 2(\partial_1\bar{\rho}_\xi\partial_1\bar{\rho}_\eta)) + 2x^1\lambda(W_{\beta,\beta} + \tilde{W}_{\beta,\beta}W_{-\beta,-\beta})\big]. \tag{14}$$

Here auxiliary vertex operators are defined as

$$\begin{aligned} W_{a,b}(x) &=: \exp\big(ia\rho_\phi(x) + ib\bar{\rho}_\phi(x)\big): \\ &= \exp\big(ia\rho_\phi^{(-)} + ib\bar{\rho}_\phi^{(-)}\big)\exp\big(ia\rho_\phi^{(+)} + ib\bar{\rho}_\phi^{(+)}\big) \\ \tilde{W}_{a,b}(x) &=: \exp(ia\rho_\eta(x) + ib\bar{\rho}_\eta(x)): \\ &= \exp\big(ia\rho_\eta^{(-)} + ib\bar{\rho}_\eta^{(-)}\big)\exp(ia\rho_\eta^{(+)} + ib\bar{\rho}_\eta^{(+)}) \end{aligned} \tag{15}$$

where $\rho_\phi^{(-)}, \rho_\eta^{(-)}$ and $\rho_\phi^{(+)}, \rho_\eta^{(+)}$ are the creation part and annihilation part of the corresponding chiral operators, respectively. For example,

$$\begin{aligned} \rho_\phi^{(+)} &= \tfrac{1}{2}\int_{-\infty}^{+\infty} dk\, a_k\left[\frac{1}{\sqrt{|k|}}e^{ikx^1} - i\sqrt{|k|}\int_{-\infty}^{x^1} dy\, e^{iky}\right] \\ \rho_\phi^{(-)} &= \tfrac{1}{2}\int_{-\infty}^{+\infty} dk\, a_k^+\left[\frac{1}{\sqrt{|k|}}e^{-ikx^1} + i\sqrt{|k|}\int_{-\infty}^{x^1} dy\, e^{-iky}\right]. \end{aligned} \tag{16}$$

Using equation (9), we get

$$\begin{aligned} \big[\rho_\phi^{(+)}(x), \rho_\phi^{(-)}(y)\big] &= -\ln\big[ik_0(x^1 - y^1 - i\epsilon)\big] \\ \big[\bar{\rho}_\phi^{(+)}(x), \bar{\rho}_\phi^{(-)}(y)\big] &= -\ln\big[-ik_0(x^1 - y^1 + i\epsilon)\big] \\ \big[\rho_\phi^{(\pm)}(x), \bar{\rho}_\phi^{(\mp)}(y)\big] &= -i\tfrac{1}{2}\pi \\ \big[\rho_\xi^{(\pm)}(x), \rho_\eta^{(\mp)}(y)\big] &= \mp\ln\big[\pm ik_0(x^1 - y^1 \mp i\epsilon)\big] \\ \big[\rho_\xi^{(\pm)}(x), \bar{\rho}_\eta^{(\mp)}(y)\big] &= -i\tfrac{1}{2}\pi \\ \big[\bar{\rho}_\xi^{(\pm)}(x), \bar{\rho}_\eta^{(\mp)}(y)\big] &= \mp\ln\big[\mp ik_0(x^1 - y^1 \pm i\epsilon)\big] \\ \big[\bar{\rho}_\xi^{(\pm)}(x), \rho_\eta^{(\mp)}(y)\big] &= i\tfrac{1}{2}\pi \end{aligned} \tag{17}$$

with the remainder vanishing. In (17) the factor k_0 ($k_0 \to 0$) comes from introducing an infrared cut-off $k_0e^{-\gamma}$ into the k-integrals when we compute these commutators, with γ the Euler constant.

We can obtain the following useful relations from (17):

$$\begin{aligned} [W_{a,b}(x), W_{c,d}(y)] = k_0^{ac+bd}\big(&i^{(a-b)(c+d)}(x^1 - y^1 - i\epsilon)^{ac}(x^1 - y^1 + i\epsilon)^{bd} \\ &- i^{-(a-b)(c+d)}(x^1 - y^1 + i\epsilon)^{ac}(x^1 - y^1 - i\epsilon)^{bd}\big) \\ &: \exp(ia\rho_\phi(x) + ib\bar{\rho}_\phi(x) + ic\rho_\phi(y) + id\bar{\rho}_\phi(y)): \end{aligned} \tag{18}$$

$$[W_{a,b}(x),\,\tilde W_{c,d}(y)]=0$$
$$[\tilde W_{a,b}(x),\,\tilde W_{c,d}(y)]=0$$

and

$$\begin{aligned}
&\mathrm{i}\partial_- W_{a,0}(x)=\frac{\lambda}{2\pi}\int \mathrm{d}y^1[W_{a,0}(x),\,W_{\beta,\beta}+\tilde W_{\beta,\beta}(y)W_{-\beta,-\beta}(y)]\\
&\mathrm{i}\partial_+ W_{0,b}(x)=\frac{\lambda}{2\pi}\int \mathrm{d}y^1[W_{0,b}(x),\,W_{\beta,\beta}+\tilde W_{\beta,\beta}(y)W_{-\beta,-\beta}(y)]\\
&\partial_-\tilde W_{a,0}=\partial_+\tilde W_{0,b}=0\,.
\end{aligned}\tag{19}$$

Furthermore,

$$\begin{aligned}
[W_{a,b}(x),\,M]&=\mathrm{i}\big(x^0\partial_1+x^1\partial_0+\tfrac12\big(a^2-b^2\big)\big)W_{a,b}(x)\\
&\quad-\frac{\lambda}{2\pi}\int \mathrm{d}y^1\,(x^1-y^1)[W_{a,b}(x),\,W_{\beta,\beta}+\tilde W_{\beta,\beta}(y)W_{-\beta,-\beta}(y)]\\
[\tilde W_{a,b}(x),\,M]&=\mathrm{i}(x^0\partial_1+x^1\partial_0)\tilde W_{a,b}(x)\,.
\end{aligned}\tag{20}$$

Having these relations, we can conveniently evaluate the non-local conserved currents of this system in a non-perturbative way.

Non-local conserved currents of this model

Taking account of

$$\lim_{\epsilon\to 0}\left(\left(x^1-y^1-\mathrm{i}\epsilon\right)^{-n}-\left(x^1-y^1+\mathrm{i}\epsilon\right)^{-n}\right)=2\pi\mathrm{i}\frac{(-1)^{n-1}}{(n-1)!}\delta^{(n-1)}\left(x^1-y^1\right)\tag{21}$$

where $\delta^{(n)}(x)$ stands for the nth derivative of $\delta(x)$, we get that, from (18) and (19),

$$\partial_- W_{-2/\beta,0}(x)=-\frac{\lambda}{2k_0^2}\left(\frac{\beta^2}{\beta^2-2}\partial_+ W_{\beta-2/\beta,\beta}(x)-\partial_- W_{\beta-2/\beta,\beta}(x)-\frac{2}{\beta^2-2}X_1\right)\tag{22}$$

and

$$\begin{aligned}
\partial_-(\tilde W_{-2/\beta,0}(x)W_{2/\beta,0}(x))=-\frac{\lambda}{2k_0^2}&\left(\frac{\beta^2}{\beta^2-2}\partial_+(\tilde W_{\beta-2/\beta,\beta}(x)W_{2/\beta-\beta,-\beta}(x))\right.\\
&\left.-\partial_-(\tilde W_{\beta-2/\beta,\beta}(x)W_{2/\beta-\beta,-\beta}(x))+\frac{2}{\beta^2-2}X_2\right)
\end{aligned}\tag{23}$$

where

$$\begin{aligned}
X_1=-\lambda k_0^{2(\beta^2-1)}&\int \mathrm{d}y^1\left(\left(x^1-y^1\right)^2+\epsilon^2\right)^{\beta^2}\delta^{(1)}\left(x^1-y^1\right)\\
&:\mathrm{e}^{\mathrm{i}(\beta-2/\beta)\rho_\phi(x)+\mathrm{i}\beta\bar\rho_\phi(x)+\mathrm{i}\beta\rho_\phi(y)+\mathrm{i}\beta\bar\rho_\phi(y)}:\\
&+\lambda k_0^{2(1-\beta^2)}\int \mathrm{d}y^1\left(\left(x^1-y^1\right)^2+\epsilon^2\right)^{(2-\beta^2)}\delta^{(1)}\left(x^1-y^1\right)\tilde W_{\beta,\beta}(y)\\
&:\mathrm{e}^{\mathrm{i}(\beta-2/\beta)\rho_\phi(x)+\mathrm{i}\beta\bar\rho_\phi(x)-\mathrm{i}\beta\rho_\phi(y)-\mathrm{i}\beta\bar\rho_\phi(y)}:\\
X_2=\tfrac12\lambda k_0^{-2\beta^2}\tilde W_{\beta-2/\beta,\beta}(x)&\int \mathrm{d}y^1\,((x^1-y^1)^2+\epsilon^2)^{(2-\beta^2)}\delta^{(1)}(x^1-y^1)\\
&:\mathrm{e}^{\mathrm{i}(2/\beta-\beta)\rho_\phi(x)-\mathrm{i}\beta\bar\rho_\phi(x)+\mathrm{i}\beta\rho_\phi(y)+\mathrm{i}\beta\bar\rho_\phi(y)}:\\
&+\tfrac12\lambda^2k_0^{-2(2+\beta^2)}\tilde W_{\beta-2/\beta,\beta}(x)\int \mathrm{d}y^1\,\left(\left(x^1-y^1\right)^2+\epsilon^2\right)^{\beta^2}\delta^{(1)}\left(x^1-y^1\right)\tilde W_{\beta,\beta}(y)\\
&:\mathrm{e}^{\mathrm{i}(2/\beta-\beta)\rho_\phi(x)-\mathrm{i}\beta\bar\rho_\phi(x)-\mathrm{i}\beta\rho_\phi(y)-\mathrm{i}\beta\bar\rho_\phi(y)}:\,.
\end{aligned}$$

To ensure (22) and (23) behave as current conservation equations, we have to demand X_1 and X_2 to be vanishing. Upon integrating $\delta^{(1)}(x^1 - y^1)$ by parts, X_1 and X_2 vanish when the powers of $(x^1 - y^1)$ are positive, which leads to a restriction on the values of the coupling constant β

$$2(2-\beta^2) - 1 > 0 \qquad 2\beta^2 - 1 > 0 .$$

That is to say

$$\begin{cases} X_1 = 0 \\ X_2 = 0 \end{cases} \qquad \text{if and only if} \quad \tfrac{1}{2} < \beta^2 < \tfrac{3}{2} . \tag{24}$$

In the region $\frac{1}{2} < \beta^2 < \frac{3}{2}$, we get four non-local conserved CR currents for the conformal $\widehat{sl}(2)$ Toda system (1)

$$\begin{aligned} j_+^{(-)}(x) &= W_{-2/\beta,0}(x) - \tfrac{1}{2}\lambda k_0^{-2} W_{\beta-2/\beta,\beta}(x) \\ j_-^{(-)}(x) &= \tfrac{1}{2}\lambda k_0^{-2} \frac{\beta^2}{\beta^2 - 2} W_{\beta-2/\beta,\beta}(x) \end{aligned} \tag{25}$$

$$\begin{aligned} j_+^{(+)}(x) &= \tilde{W}_{-2/\beta,0}(x) W_{2/\beta,0}(x) - \tfrac{1}{2}\lambda k_0^{-2} \tilde{W}_{\beta-2/\beta,\beta}(x) W_{\beta-2/\beta,-\beta}(x) \\ j_-^{(+)}(x) &= \tfrac{1}{2}\lambda k_0^{-2} \frac{\beta^2}{\beta^2 - 2} \tilde{W}_{\beta-2/\beta,\beta}(x) W_{2/\beta-\beta,-\beta}(x) \end{aligned} \tag{26}$$

$$\begin{aligned} \tilde{j}_+^{(-)}(x) &= \tfrac{1}{2}\lambda k_0^{-2} \frac{\beta^2}{\beta^2 - 2} W_{\beta,\beta-2/\beta}(x) \\ \tilde{j}_-^{(-)}(x) &= W_{0,-2/\beta}(x) - \tfrac{1}{2}\lambda k_0^{-2} W_{\beta,\beta-2/\beta}(x) \end{aligned} \tag{27}$$

$$\begin{aligned} \tilde{j}_+^{(+)}(x) &= \tfrac{1}{2}\lambda k_0^{-2} \frac{\beta^2}{\beta^2 - 2} \tilde{W}_{\beta,\beta-2/\beta}(x) W_{-\beta,2/\beta-\beta}(x) \\ \tilde{j}_-^{(+)}(x) &= \tilde{W}_{0,-2/\beta}(x) W_{0,2/\beta}(x) - \tfrac{1}{2}\lambda k_0^{-2} \tilde{W}_{\beta,\beta-2/\beta}(x) W_{-\beta,2/\beta-\beta}(x) . \end{aligned} \tag{28}$$

They satisfy the following conservation equations:

$$\partial_+ j_-^{(\pm)}(x) + \partial_- j_+^{(\pm)}(x) = 0 \qquad \partial_+ \tilde{j}_-^{(\pm)}(x) + \partial_- \tilde{j}_+^{(\pm)}(x) = 0 . \tag{29}$$

These CR currents are defined non-perturbatively, whose expressions differ from the corresponding BL currents [2] in the presence of extra terms $-\frac{1}{2}\lambda k_0^{-2} W_{\beta-2/\beta,\beta}(x)$ in $j_+^{(-)}(x)$, $-\frac{1}{2}\lambda k_0^{-2} \tilde{W}_{\beta-2/\beta,\beta}(x)$ in $j_+^{(+)}(x)$, $-\frac{1}{2}\lambda k_0^{-2} W_{\beta,\beta-2/\beta}(x)$ in $\tilde{j}_-^{(-)}(x)$ and $-\frac{1}{2}\lambda k_0^{-2} \tilde{W}_{\beta,\beta-2/\beta}(x) W_{-\beta,2/\beta-\beta}(x)$ in $\tilde{j}_-^{(+)}(x)$. Nevertheless, the equal-time commutation relations of CR charges for conformal $\widehat{sl}(2)$ Toda system are always in accordance with their counterparts of BL charges, because the commutation relations obeyed by BL charges have proved to give non-perturbative results. If ρ_ϕ and $\bar{\rho}_\phi$ were true chiral operators, X_1 and X_2 would naturally be zero, and then there would be no restriction on the values of coupling constant β, as well as the conserved currents would take the same forms as the corresponding BL currents.

In the allowed regions of coupling $\frac{1}{2} < \beta^2 < \frac{3}{2}$, the non-local currents (25)–(28) are Lorentz covariant currents. In terms of (20), we have

$$\begin{aligned} [j_+^{(\pm)}(x), M] &= \mathrm{i}\left(x^0\partial_1 + x^1\partial_0 + \frac{2}{\beta^2}\right) j_+^{(\pm)}(x) \\ [j_-^{(\pm)}(x), M] &= \mathrm{i}\left(x^0\partial_1 + x^1\partial_0 + \left(\frac{2}{\beta^2} - 2\right)\right) j_-^{(\pm)}(x) \end{aligned} \tag{30}$$

$$[\bar{j}_+^{(\pm)}(x), M] = \mathrm{i}\left(x^0\partial_1 + x^1\partial_0 + \left(2 - \frac{2}{\beta^2}\right)\right)\bar{j}_+^{(\pm)}(x)$$
$$[\bar{j}_-^{(\pm)}(x), M] = \mathrm{i}\left(x^0\partial_1 + x^1\partial_0 - \frac{2}{\beta^2}\right)\bar{j}_-^{(\pm)}(x). \tag{31}$$

Namely the currents $(j_+^{(\pm)}, j_-^{(\pm)})$ and $(\bar{j}_+^{(\pm)}, \bar{j}_-^{(\pm)})$ do carry Lorentz weights $(2/\beta^2, 2/\beta^2 - 2)$ and $(2 - 2/\beta^2, -2/\beta^2)$, respectively. The difference of 2 in their weights between the current components is just what is needed to make the current conservation equations (29) covariant under Lorentz transformation.

The charges corresponding to above non-local currents are defined as

$$Q_- \equiv \tfrac{1}{2}\int \mathrm{d}x^1\,(j_+^{(-)}(x) + j_-^{(-)}(x)) \qquad Q_+ \equiv \tfrac{1}{2}\int \mathrm{d}x^1\,(j_+^{(+)}(x) + j_-^{(+)}(x))$$
$$\bar{Q}_+ \equiv \tfrac{1}{2}\int \mathrm{d}x^1\,(\bar{j}_+^{(-)}(x) + \bar{j}_-^{(-)}(x)) \qquad \bar{Q}_- \equiv \tfrac{1}{2}\int \mathrm{d}x^1\,(\bar{j}_+^{(+)}(x) + \bar{j}_-^{(+)}(x)). \tag{32}$$

Using the fundamental commutation relations (17), one can verify that these charges obey the algebra

$$Q_\pm\bar{Q}_\pm - q^2\bar{Q}_\pm Q_\pm = 0$$
$$Q_+\bar{Q}_- - q^{-2}\bar{Q}_-Q_+ = a\left(1 - q^{2J+2J_\eta}\right) \tag{33}$$
$$Q_-\bar{Q}_+ - q^{-2}\bar{Q}_+Q_- = a\left(1 - q^{-2J}\right)$$

where

$$J = \frac{\beta}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}x^1\left(\frac{\partial\rho_\phi}{\partial x^1} + \frac{\partial\bar{\rho}_\phi}{\partial x^1}\right) \qquad J_\eta = -\frac{\beta^3}{4\pi}\int_{-\infty}^{+\infty}\mathrm{d}x^1\left(\frac{\partial\rho_\eta}{\partial x^1} + \frac{\partial\bar{\rho}_\eta}{\partial x^1}\right)$$

are two topological charges and

$$q = \mathrm{e}^{-2\pi\mathrm{i}/\beta^2} \qquad a = -\mathrm{i}\pi\frac{\lambda}{k_0^2}\left(\frac{1}{q(\beta^2-2)^2}\right).$$

This charge algebra is really in agreement with the analogue given by BL, except the constant coefficient 'a' (BLs $a = -\mathrm{i}(\lambda/2\pi)((\beta^2)^2/(\beta^2-2)^2)$). The difference of the coefficient is unimportant, because one can redefine charges $Q' = \sqrt{q/2}(k_0\beta^2/\pi)Q$ so that the charge algebra is the same as the BLs.

Conclusion and remarks

A set of non-local Lorentz covariant conserved currents have been obtained non-perturbatively for the quantum $\widehat{sl}(2)$ Toda model, and these currents exist, only when the coupling constant β takes some special values. Our currents differ greatly from BLs which are the perturbative-dependent and can take any coupling constant β, because the chiral operators (10) is not a real chiral function of spacetime in our non-perturbation case. But, *the quantum group algebra satisfied by these conserved charges is the same as that satisfied* by the BL charges. Moreover, when ρ limits chiral operators, there is no difference between our currents and the BL currents. When $\eta \to 0$, i.e. the $\widehat{sl}(2)$ Toda model degenerates to the Sine–Gordon model, we naturally get the CRs results, which means our outcome is an extension of CRs. We have to point out that the method discussed here is not suitable for the non-simply-laced affine Toda and non-simply-laced conformal affine Toda models, because there are only two (not four) conserved charges arising in the non-perturbative scheme discussed above for either of such models [10].

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